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ON THE STABILITY OF A THREE-LAYERED PLATE

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Abstract—An exact solution is presented for the three-dimensional stability problem of a thick three-layered plate with Navier boundary conditions ('simply supported' edges). The linearized equations of nonlinear theory of elasticity are considered. The conditions of the absence of slip between layers are used. For several numerical examples a comparison is made for exact solution and some approximate solutions.

1

THE stability problem for elastic body in the absence of body forces is essentially [1, 2] the eigenvalue and eigenfunction boundary problem for the equations of nonlinear elasticity theory, which are linearized for perturbations.

These equations and boundary conditions are as follows:

$$\frac{\partial \sigma'_x}{\partial x}+\frac{\partial \tau'_{xy}}{\partial y}+\frac{\partial \tau'_{xz}}{\partial z}+L^{\circ}u' = 0,$$

$$\frac{\partial \tau'_{xy}}{\partial x}+\frac{\partial \sigma'_y}{\partial y}+\frac{\partial \tau'_{xz}}{\partial z}+L^{\circ}v' = 0,$$

$$\frac{\partial \tau'_{xz}}{\partial x}+\frac{\partial \tau'_{yz}}{\partial y}+\frac{\partial \sigma'_z}{\partial z}+L^{\circ}w' = 0, \tag{1}$$

$$L^{\circ} = \sigma^{\circ}_x\frac{\partial^2}{\partial x^2}+\sigma^{\circ}_y\frac{\partial^2}{\partial y^2}+\sigma^{\circ}_z\frac{\partial^2}{\partial z^2}+2\tau^{\circ}_{xy}\frac{\partial^2}{\partial x\partial y}+2\tau^{\circ}_{xz}\frac{\partial^2}{\partial x\partial z}+2\tau^{\circ}_{yz}\frac{\partial^2}{\partial y\partial z}$$

$$\left[\sigma'_x+\sigma^{\circ}_x\frac{\partial u'}{\partial x}+\tau^{\circ}_{xy}\frac{\partial u'}{\partial y}+\tau^{\circ}_{xz}\frac{\partial u'}{\partial z}\right]\cos\hat{nx}$$
$$+\left[\tau'_{xy}+\tau^{\circ}_{xy}\frac{\partial u'}{\partial x}+\sigma^{\circ}_y\frac{\partial u'}{\partial y}+\tau^{\circ}_{yz}\frac{\partial u'}{\partial z}\right]\cos\hat{ny}$$
$$+\left[\tau'_{xz}+\tau^{\circ}_{xz}\frac{\partial u'}{\partial x}+\tau^{\circ}_{yz}\frac{\partial u'}{\partial y}+\sigma^{\circ}_z\frac{\partial u'}{\partial z}\right]\cos\hat{nz} = S'_x, \tag{2}$$

The subscript 'o' corresponds to the unperturbed stress field. The latter, by assumption, may be determined from the equations of linear elasticity theory. The primes correspond to the perturbations.

Using Hooke's law for perturbations we obtain a set of equations

$$(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)+\mu\Delta u'+L^{o}v' = 0,$$

$$(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)+\mu\Delta u'+L^{o}v' = 0, \tag{3}$$

$$(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)+\mu\Delta w'+L^{o}w' = 0.$$

2

Consider stability of equilibrium of rectangular elastic plate which is composed of three isotropic layers. The plate is assumed to be symmetric with respect to the z-plane (Fig. 1), elastic coefficients λ_1, μ_1 identical for two external layers and Poisson ratio ν

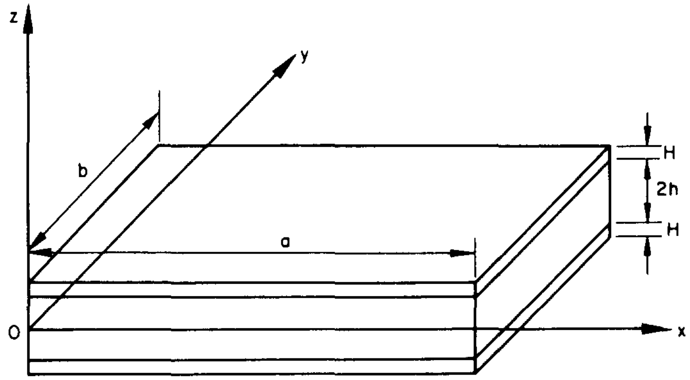

Fig. 1.

identical for all layers. Moreover, we assume the loads p, p_1 and q, q_1 to be connected by the relationship:

$$\frac{p}{\mu} = \frac{p_1}{\mu_1}, \qquad \frac{q}{\mu} = \frac{q_1}{\mu_1}. \tag{4}$$

Here λ, μ are Lamé elastic constants for the internal layer.

Then unperturbed stress field is determined by equations

$$\sigma_x^o = -p, \qquad \sigma_y^o = -q, \qquad \sigma_z^o = \tau_{xy}^o = \tau_{xz}^o = \tau_{yz}^o = 0, \tag{5}$$

for the internal layer, and

$$\sigma_x^o = -p_1, \qquad \sigma_y^o = -q_1, \qquad \sigma_z^o = \tau_{xy}^o = \tau_{xz}^o = \tau_{yz}^o = 0. \tag{6}$$

for the external layers.

The boundary conditions on planes $x = 0$, $x = a$, $y = 0$, $y = b$ are (here and in the following the primes are dropped)

$$\sigma_n+\sigma_n^o\frac{\partial u_n}{\partial n} = 0, \qquad u_s = 0, \qquad w = 0. \tag{7}$$

Here σ_n is normal stress, u_n is normal displacement, u_1 is tangential displacement in plane xy. The boundary conditions (7) correspond to the Navier ones, which are used in the theory of thin plates.

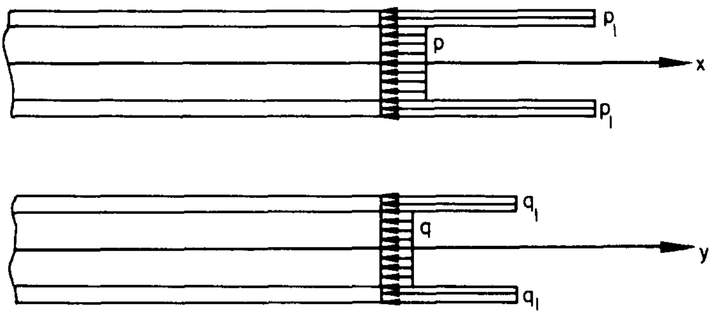

FIG. 2.

The conditions of the absence of slip between layers are

$$u|_{z=h-0} = u|_{z=h+0}; \qquad v|_{z=h-0} = v|_{z=h+0}; \qquad w|_{z=h-0} = w|_{z=h+0}. \tag{8}$$

Equating stress components on contact surface, we obtain

$$S_x|_{z=h-0} = S_x|_{z=h+0}; \qquad S_y|_{z=h-0} = S_y|_{z=h+0}; \qquad S_z|_{z=h-0} = S_z|_{z=h+0}. \tag{9}$$

Using relations (9) and (2) we obtain conditions for stress components

$$\tau_{xz}|_{z=h-0} = \tau_{xz}|_{z=h+0}; \qquad \tau_{yz}|_{z=h-0} = \tau_{yz}|_{z=h+0}; \qquad \sigma_z|_{z=h-0} = \sigma_z|_{z=h+0}. \tag{10}$$

Equating stresses on the external surfaces to the zero, we obtain

$$\tau_{xz}|_{z=h+H} = 0, \qquad \tau_{yz}|_{z=h+H} = 0, \qquad \sigma_z|_{z=h+H} = 0. \tag{11}$$

3

We shall look for a solution of equations (3) in the form

$$u = F(z)\cos k_1x \sin k_2y, \qquad v = \Phi(z)\sin k_1x\cos k_2y, \qquad w = \Psi(z)\sin k_1x\sin k_2y, \tag{12}$$

$$k_1 = \frac{m_1\pi}{a}, \qquad k_2 = \frac{m_2\pi}{b}\ (m_{1,2} = 1, 2, \ldots)$$

Substituting relations (12) in (3), we obtain a set of equations for functions F, Φ, Ψ

$$\begin{gathered}\mu F'' - [(\lambda+\mu)k_1^2 + (1-\alpha)\mu k^2]F - (\lambda+\mu)k_1k_2\Phi + (\lambda+\mu)k_1\Psi' = 0,\\ -(\lambda+\mu)k_1k_2F + \mu\Phi'' - [(\lambda+\mu)k_2^2 + (1-\alpha)\mu k^2]\Phi + (\lambda+\mu)k_2\Psi' = 0,\\ -(\lambda+\mu)k_1F' - (\lambda+\mu)k_2\Phi' + (\lambda+2\mu)\Psi'' - (1-\alpha)\mu k^2\Psi = 0,\\ \left(k^2 = k_1^2 + k_2^2,\ \alpha = \frac{pk_1^2 + qk_2^2}{\mu k^2}\right).\end{gathered} \tag{13}$$

Roots of the corresponding characteristic equation are given by

$$\gamma_{1,2} = \pm\ell_1, \qquad \gamma_{3,4} = \pm\ell_2, \qquad \gamma_{5,6} = \pm\ell_3.$$

Here

$$\ell_1^2 = \ell_2^2 = k^2(1-\alpha), \qquad \ell_3^2 = k^2(1-\tau\alpha), \qquad \tau = \frac{\mu}{\lambda+2\mu}.$$

The general solution of equations (13) is

$$\begin{aligned}
F(z) &= (-k_2a_1+k_1a_2)\,\mathrm{sh}\,\ell_1z+k_1a_3\,\mathrm{sh}\,\ell_3z+(-k_2b_1+k_1b_2)\,\mathrm{ch}\,\ell_1z+k_1b_3\,\mathrm{ch}\,\ell_3z,\\
\Phi(z) &= (k_1a_1+k_2a_2)\,\mathrm{sh}\,\ell_1z+k_2a_3\,\mathrm{sh}\,\ell_3z+(k_1b_1+k_2b_2)\,\mathrm{ch}\,\ell_1z+k_2b_3\,\mathrm{ch}\,\ell_3z,\\
\Psi(z) &= \frac{k^2}{\ell_1}a_2\,\mathrm{ch}\,\ell_1z+\ell_3a_3\,\mathrm{ch}\,\ell_3z+\frac{k^2}{\ell_1}b_2\,\mathrm{sh}\,\ell_1z+\ell_3b_3\,\mathrm{sh}\,\ell_3z.
\end{aligned} \tag{14}$$

Here a_{1-3}, b_{1-3} are integration constants.

In order to satisfy conditions (10) and (11), we must determine components τ_{xz}, τ_{yz}, σ_z of stress tensor of perturbations:

$$\begin{aligned}
\tau_{xz} = \mu\Bigg\{&\left(-k_2a_1\ell_1+k_1\frac{k^2+\ell_1^2}{\ell_1}a_2\right)\mathrm{ch}\,\ell_1z+2k_1\ell_3a_3\,\mathrm{ch}\,\ell_3z\\
&+\left(-k_2b_1\ell_1+k_1\frac{k^2+\ell_1^2}{\ell_1}b_2\right)\mathrm{sh}\,\ell_1z+2k_1\ell_3b_3\,\mathrm{sh}\,\ell_3z\Bigg\}\cos k_1x\sin k_2y,\\
\tau_{yz} = \mu\Bigg\{&\left(k_1\ell_1a_1+k_2\frac{k^2+\ell_1^2}{\ell_1}a_2\right)\mathrm{ch}\,\ell_1z+2k_2\ell_3a_3\,\mathrm{ch}\,\ell_3z\\
&+\left(k_1\ell_1b_1+k_2\frac{k^2+\ell_1^2}{\ell_1}b_2\right)\mathrm{sh}\,\ell_1z+2k_2\ell_3b_3\,\mathrm{sh}\,\ell_3z\Bigg\}\sin k_1x\cos k_2y,\\
\sigma_z = \mu\{&2k^2a_2\,\mathrm{sh}\,\ell_1z+(k^2+\ell_1^2)a_3\,\mathrm{sh}\,\ell_3z+2k^2b_2\,\mathrm{ch}\,\ell_1z+(k^2+\ell_1^2)b_3\,\mathrm{ch}\,\ell_3z\}\sin k_1x\sin k_2y.
\end{aligned}$$

Solutions for the external layers have similar form. One must substitute μ_1, A_i, B_i for the upper layer, and μ_1, A_i', B_i' for the lower layer instead of μ, a_i, b_i.

4

It follows from the symmetry of the plate with respect to the z-plane, and from the symmetry of conditions on the contact surfaces and the edge conditions, that two modes of buckling are possible. The first mode is antisymmetric (buckling with bending), the second one corresponds to buckling, which is symmetric with respect to the z-plane.

We obtain for antisymmetric modes

$$b_i = 0, \qquad A_i' = A_i, \qquad B_i' = B_i \qquad (i = 1, 2, 3). \tag{15}$$

After satisfying equations (8), (10) and (11), we obtain nine equations for nine unknown constants a_i, A_i, B_i. Critical values of load parameter α correspond to zeros of determinant of this set of equations. The determinant may be represented as the product of two terms. In this connection we consider two cases.

(1) First factor is equal to zero:

$$\xi c_1C_1+t_1^2s_1S_1 = 0. \tag{16}$$

In this case we have

$$a_2 = a_3 = 0, \qquad A_2 = A_3 = 0, \qquad B_2 = B_3 = 0.$$

The w-component vector disappears.

(2) Second factor is equal to zero:

$$\begin{aligned}&\xi\alpha^2[(\zeta^2 C_1 S_3 - 4t_1^2 S_1 C_3)c_1 c_3 + (\zeta^2 S_1 C_3 - 4t_3^2 C_1 S_3)t_1^2 s_1 s_3] + \xi^2\alpha^2(\zeta^2 c_1 s_3 - 4t_1^2 s_1 c_3)\\ &\quad + [\zeta^2(C_1 C_3 - 1) - 4t_1^2 t_3^2 S_1 S_3][(2-\xi\zeta)^2 c_1 s_3 - 4(1-\xi)^2 t_1^2 s_1 c_3]\\ &\quad + [\zeta^2 S_1 S_3 - 4(C_1 C_3 - 1)][(\zeta - 2\xi)^2 t_1^2 s_1 c_3 - \zeta^2(1-\xi)^2 c_1 s_3] = 0.\end{aligned} \tag{17}$$

Here

$$\xi = \frac{\mu}{\mu_1}, \qquad t_i = \frac{\dot{t}_i}{k}, \qquad \zeta = 1 + t_1^2, \qquad s_i = \frac{\operatorname{sh} \dot{t}_i h}{t_i},$$

$$c_i = \operatorname{ch} \dot{t}_i h, \qquad S_i = \frac{\operatorname{sh} \dot{t}_i H}{t_i}, \qquad C_i = \operatorname{ch} \dot{t}_i H.$$

In this case the constants a_1, A_1, B_1 must be set equal to zero.

For symmetric deformation we have relations

$$a_i = 0, \qquad A_i = -A_i, \qquad B_i' = B_i \qquad (i = 1, 2, 3). \tag{18}$$

Satisfying boundary conditions (8), (10) and (11) and equating to zero the determinant of the resulting system, we obtain the equation for critical value of loading parameter. Just as before the determinant may be represented as the product of two terms, and one must consider two cases.

(1) First factor is equal to zero:

$$\xi C_1 s_1 + S_1 c_1 = 0. \tag{19}$$

Then we have

$$b_2 = b_3 = 0, \qquad A_2 = A_3 = 0, \qquad B_2 = B_3 = 0. \tag{20}$$

For such a type of strain field the displacement in the z-direction is absent.

(2) Equating the second factor to zero yields:

$$\begin{aligned}&\xi\alpha^2[t_3^2 s_1 s_3(\zeta^2 C_1 S_3 - 4t_1^2 C_3 S_1) + c_1 c_3(\zeta^2 S_1 C_3 - 4t_3^2 C_1 S_3)] + \xi^2\alpha^2(\zeta^2 s_1 c_3 - 4t_3^2 c_1 s_3)\\ &\quad + [\zeta^2(C_1 C_3 - 1) - 4t_1^2 t_3^2 S_1 S_3][(2-\xi\zeta)^2 s_1 c_3 - 4(1-\xi)^2 t_3^2 c_1 s_3]\\ &\quad + [\zeta^2 S_1 S_3 - 4(C_1 C_3 - 1)][(\zeta - 2\xi)^2 t_3^2 c_1 s_3 - \zeta^2(1-\xi)^2 s_1 c_3] = 0.\end{aligned} \tag{21}$$

Note, that of all critical values of α only those which are small compared with unity are physically possible. This is due to the fact that in the original strain field the strains are of the order p/μ or q/μ (i.e. α), whereas the original equations are based on the assumption that these strains are small.

It must be noted also, that assumption (4) and the assumption about constant value of Poisson's ratio may be removed. This removal will lead only to different expressions for the roots of characteristic equations for internal and external layers; equations (18) and (21) would be slightly more complex.

5

Consider antisymmetric (with bending) buckling of a thick square three-layered plate with a soft internal layer ($a = b = 10(h+H)$, $H = 0{\cdot}1(h+H)$, $\nu = 0{\cdot}3$, $\xi = 0{\cdot}001$). An approximate approach [3, 4] gives $\alpha = 0{\cdot}00492$. From equation (17) we obtain $\alpha = 0{\cdot}00497$.

For the plate with a rigid internal layer ($a = b = 10(h+H)$, $H = 0{\cdot}5(h+H)$, $\nu = 0{\cdot}3$, $\xi = 0{\cdot}1$) an approximate theory [5] gives the value $\alpha = 0{\cdot}0599$, while the exact value is found to be $\alpha = 0{\cdot}0631$.

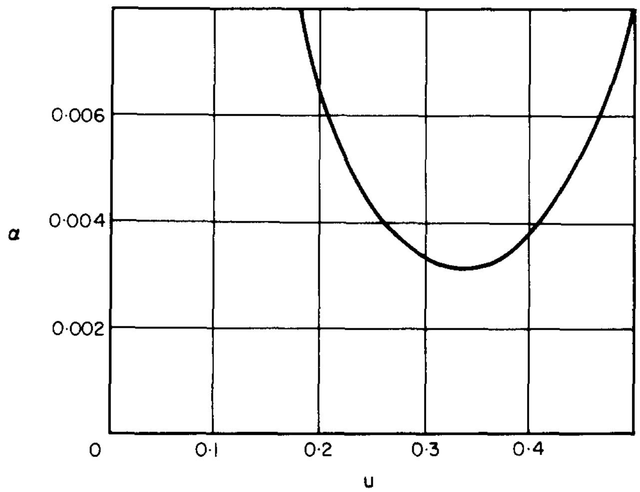

FIG. 3.

The exact values of α for symmetric buckling are given in Fig. 3 vs. ratio 'u' of plate half-thickness $(h+H)$ to length and half-wave of buckling

$$a/m_1 = b/m_2 \, (\nu = 0{\cdot}4, \; \xi = 0{\cdot}000119).$$

The minimum value is $\alpha = 0{\cdot}00330$. Approximate approaches [3, 4] yield $\alpha = 0{\cdot}00273$. Mushtari's approximate theory [6] yields $\alpha = 0{\cdot}00320$. The critical value of α based on the assumption that the external layers are plates governed by the Kirchhoff–Love theory and the internal layer is an elastic continuum media (parametric terms are not taken into account) [6] is found to be $\alpha = 0{\cdot}00315$.

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Zusammenfassung—Für eine dicke, dreischichtige Platte mit Navier'schen Randbedingungen ('einfach gestützte' Kanten) wird eine genaue Lösung des dreidimensionalen Stabilitätsproblems angegeben. Die linearisierten Gleichungen der nichtlinearen Elastizitätstheorie werden in Betracht gezogen. Verwendet werden die bei der Abwesenheit von Zwischenschichtschlupf bestehenden Bedingungen. Anhand mehrerer Zahlenbeispiele wird die genaue Lösung mit einigen Annäherungslösungen verglichen.

Абстракт—Предлагается точное решение в трехмерной постановке задачи об устойчивости толстой трехслойной плиты при граничных условиях Навье (''опертый'' край). Рассматриваются линеаризированные уравнения нелинейной теории упругости. Используются условия отсутствия сдвига между слоями. На нескольких численных примерах дается сопоставление точного решения и некоторых приближенных решений.